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An Ore-type condition for pancyclability

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Abstract

Let G be a graph of order n and S a subset of $V(G)$. We define G to be *S-pancyclable* if for every integer $l, 3 \leqslant l \leqslant |S|$, there exists a cycle in G that contains exactly l vertices of S. We prove that if the degree sum in G of every pair of nonadjacent vertices of S is at least n, then G is either S-pancyclable or else n is even, $S = V(G)$ and $G = K_{n/2,n/2}$, or $|S| = 4$, $G[S] = K_{2,2}$ and the structure of G is well characterized.

Keywords: Graph; Pancyclic; Cyclable

We consider only finite undirected graphs without loops or multiple edges. Given a graph G, we denote by $V(G), E(G)$, respectively, the sets of vertices and edges of G. For $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S. For $x \in V(G)$, $N_S(x) = \{v \in S : vx \in E(G)\}$ and $d_S(x) = |N_S(x)|$; if there is no ambiguity, we write $N(x)$ for $N_G(x)$ and $d(x)$ for $d_G(x)$.

For a cycle C in G with a given orientation and X a subset of $V(C)$, X^+ (respectively, X^-) is the set of the successors (predecessors, respectively) of the vertices of X in C, and for a and b in C, we define $C[a,b]$ ($C[a,b)$, $C(a,b)$, respectively) to be the subpath of C from a to b (from a to b^-, from a^+ to b^-, respectively). Other notation can be found in [2].

Let S be a subset of $V(G)$. A vertex v is called an *S-vertex* if $v \in S$. People have given different definitions and results about cycles containing certain subsets of vertices, see for example [8]. Following [4,6], the set S of vertices is called *cyclable* in G if all vertices of S belong to a common cycle in G. The *S-length* of a cycle in G is defined as the number of S-vertices that it contains and the graph G is said *S-pancyclable* if

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it contains cycles of all S-lengths from 3 to $|S|$. Obviously, if G is $V(G)$-pancyclable, then G is pancyclic, i.e. contains cycles of every length between 3 and $|V(G)|$. For this reason we choose not to require the existence of cycles of G through exactly one or two vertices of S in the definition of the S-pancyclicity. The note after Theorem 3 gives another justification for this choice.

Some results concerning cyclability of graphs can be found in [4,6]. The following theorem can be obtained as a corollary of a theorem of Ota in [6] or a theorem of Favaron et al. in [4] (if the graph is 2-connected).

Theorem 1. *Let G be a graph of order n and S a subset of $V(G)$ with $|S|\geqslant 3$. If $d(x)+d(y)\geqslant n$ for every pair of nonadjacent vertices x and y in S, then S is cyclable in G.*

Theorem 1 is similar to the well-known Ore-condition [5] that implies hamiltonicity and also the existence of cycles of every length between 3 and $|V(G)|$ as proved by Bondy in [1].

Theorem 2 (Bondy [1]). *Let G be a graph of order n. If $d(x)+d(y)\geqslant n$ for every pair of nonadjacent vertices x and y in G, then G is either pancyclic or the complete bipartite graph $K_{n/2,n/2}$.*

In [3], Bondy suggested the metaconjecture that almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (except maybe for a special family of graphs). Many results have been obtained in this problem. We follow in this paper the idea of some analogy between hamiltonicity and cyclability as well as between the notions of pancyclicity and pancyclability, and obtain the following theorem. Its proof is related to a method and result of Schmeichel and Hakimi in [7]. Clearly Theorem 2 is a corollary of Theorem 3 in the case $S=V(G)$.

Theorem 3. *Let G be a graph of order n and S a subset of $V(G)$. If $d(x)+d(y)\geqslant n$ for every pair of nonadjacent vertices x and y of S, then either G is S-pancyclable or else n is even, $S=V(G)$ and $G=K_{n/2,n/2}$, or $G[S]=K_{2,2}=C_4 := x_1x_2x_3x_4x_1$ and the structure of G is as follows: $V(G)$ is partitioned into $S\cup V_1\cup V_2\cup V_3\cup V_4$; for any i, $1\leqslant i\leqslant 4$, $G[V_i]$ is any graph on $|V_i|$ vertices with $|V_i|\geqslant 0$, and each vertex x_i is adjacent to all the vertices of V_{i+1} and V_i where the index i is taken as modulo 4.*

We do note that if $S\neq V(G)$, the hypothesis of Theorem 3 does not imply the existence of cycles of S-length 1 or 2 in G. For instance, let G be a graph of order n obtained by subdividing some edge xy of a complete graph K_{n-1} by a new vertex u, and let $S=V(G)-\{u\}$. Every cycle of G contains at least three vertices of S. Hence G does not contain any cycle of S-length 1 or 2 although the only pair $\{x,y\}$ of nonadjacent vertices of S satisfies $d(x)+d(y)>n$. Similarly, in the complete bipartite graph $K_{n/2,n/2}$ with $V = A \cup B$ and $S = A$, every cycle contains at least two vertices of S.

Proof of Theorem 3. Let $G = (V,E)$ be a graph of order n and S a subset of $V(G)$ of cardinality q that satisfies the hypothesis of Theorem 3, that is the degree sum in G of any pair of nonadjacent vertices of S is at least n. We know from Theorem 1 that G contains a cycle through all vertices of S. Choose such a cycle C with as few vertices as possible and give C some arbitrary orientation. Put $R = G - C$. Let $x_1, x_2, \dots, x_q$ be the vertices of $C \cap S$, the order 1, 2, …, q respecting the orientation of C, and consider the subscripts modulo q. Two S-vertices x_i and x_{i+1} are said to be *S-consecutive* in C and the segment $C[x_i, x_{i+1})$ is denoted by $S_i, 1 \leqslant i \leqslant q$.

Claim 1. *If the S-vertices x_i and x_j are nonadjacent and have no common neighbor in R, then* $\mathrm{d}_C(x_i) + \mathrm{d}_C(x_j) \geqslant |C|$.

Proof. Claim 1 is an easy consequence of the facts that $d(x_i) + d(x_j) \geqslant n$ and $d_R(x_i) + d_R(x_j) \leqslant |R|$. □

Assume now that G is not S-pancyclable, and more precisely that G misses a cycle of S-length l for some l, $3 \leqslant l \leqslant q-1$, l being fixed until the end of the paper (note that necessarily $|S| \geqslant 4$). From Claim 1, clearly

$$d_C(x_i) + d_C(x_{i+l-1}) \geqslant |C| \quad \text{for every } i, 1 \leqslant i \leqslant q. \tag{$*$}$$

Lemma 1. *There exists at least one pair of S-consecutive vertices in C that have degree-sum in C at least $|C|$.*

Proof. Choose $i \in \{1,2,\dots,q\}$. By (*), $d_C(x_i) + d_C(x_{i+l-1}) \geqslant |C|$ and similarly, $d_C(x_{i+1}) + d_C(x_{i+l}) \geqslant |C|$. Both inequalities cannot occur if the two S-consecutive pairs x_i, x_{i+1} and x_{i+l-1}, x_{i+l} have simultaneously degree-sum in C less than $|C|$. □

Without loss of generality, we can choose x_q and x_1 as consecutive S-vertices with maximum degree sum in C. By Lemma 1, $d_C(x_q) + d_C(x_1) \geqslant |C|$. When $x_q^+ \neq x_1$, let G' be the graph obtained from G by deleting the vertices of $C(x_q, x_1)$ and adding the edge $x_q x_1$, and let C' be the cycle of G' obtained from C by replacing the segment between x_q and x_1 by the edge $x_q x_1$. When $x_q^+ = x_1$, let $G' = G$ and $C' = C$. In C', the segments S_i remain unchanged except S_q which is transformed into $S'_q = C'[x_q, x_1) = \{x_q\}$. By the choice of C, x_q has exactly one neighbor, namely x_q^+, in $C[x_q, x_1]$. Similarly, x_1 has exactly one neighbor, namely x_1^-, in $C(x_q, x_1)$. Therefore, $d_{C'}(x_q) + d_{C'}(x_1) = d_C(x_q) + d_C(x_1)$ and thus $d_{C'}(x_q) + d_{C'}(x_1) \geqslant |C'| + \varepsilon$, where $\varepsilon = 0$ if $x_q^+ = x_1$ and $\varepsilon = 1$ if $x_q^+ \neq x_1$. On the other hand, when G' contains a cycle Γ'_l of S-length l, then G contains a cycle Γ_l of same S-length l which is obtained as follows: if $G' = G$ or if the edge $x_q x_1$ does not belong to Γ'_l, then $\Gamma_l = \Gamma'_l$; if $G' \neq G$ and Γ'_l contains the edge $x_q x_1$, then we replace the edge $x_q x_1$ by the path $C(x_q, x_1)$ of

C. By the assumption on G, we know that G' contains no cycle of S-length l. Let us work on G' now, till we get $G' = G$.

Claim 2. *Let $f_l(k) = q + k - l$ if $2 \leqslant k \leqslant l - 1$ and $f_l(k) = k - l + 2$ if $l \leqslant k \leqslant q - 1$. Then $N(x_q) \cap S_i \neq \emptyset, 2 \leqslant i \leqslant q - 1$, implies $N(x_1) \cap S^+_{f_l(i)} = \emptyset$.*

Proof. The argument is similar to that of Bondy [1]. If Claim 2 was not true, then G would contain a cycle of S-length l.

Note that $2 \leqslant f_l(k) \leqslant q - 1$ in any case and that $f_l(k)$ is one to one.

Lemma 2. (1) $d_{S_1}(x_q) + d_{S_1^+}(x_1) + d_{S'_q}(x_q) + d_{S'^+_q}(x_1) \leqslant |S_1| + |S'_q|$.

(2) *For any i, $2 \leqslant i \leqslant q - 1$,* $d_{S_i}(x_q) + d_{S_i^+}(x_1) \leqslant \begin{cases} |S_i| & \text{if } d_{S_i}(x_q) \text{ or } d_{S_i^+}(x_1) = 0 \\ |S_i| + 1 & \text{otherwise.} \end{cases}$

Proof.
(1) results from our choice of C and definition of C'.
(2) is clearly true if $i = q - 1$ or if $d_{S_i}(x_q)$ or $d_{S_i^+}(x_1) = 0$. So assume $2 \leqslant i \leqslant q - 2$ and $d_{S_i}(x_q) = t_i \neq 0$; if x_q is adjacent to $x \in S_i$, then, by the choice of C, x_1 is not adjacent to x^{++}. Hence x_1 has at least $t_i - 1$ nonadjacent vertices in S_i^+ and $d_{S_i}(x_q) + d_{S_i^+}(x_1) \leqslant t_i + |S_i^+| - (t_i - 1) = |S_i| + 1$. □

Let us come back to the proof of Theorem 3, with the same hypotheses and notation as above.

Let

$$A_1 = \{i | N(x_q) \cap S_i \neq \emptyset,\ N(x_1) \cap S_i^+ = \emptyset \text{ and } 2 \leqslant i \leqslant q - 1\},$$

$$A_2 = \{i | N(x_q) \cap S_i = \emptyset,\ N(x_1) \cap S_i^+ \neq \emptyset \text{ and } 2 \leqslant i \leqslant q - 1\}$$

and

$$A_3 = \{i | N(x_q) \cap S_i \neq \emptyset,\ N(x_1) \cap S_i^+ \neq \emptyset \text{ and } 2 \leqslant i \leqslant q - 1\}.$$

From Claim 2 we know that if $i \in A_1 \cup A_3$, then $f_l(i) \notin A_2 \cup A_3$. Thus, by the injectivity of the function f_l, $|A_1 \cup A_3| \leqslant q - 2 - |A_2 \cup A_3|$, i.e. $|A_3| \leqslant q - 2 - |A_1 \cup A_2 \cup A_3|$. By Lemma 2 we have

$$\sum_{i=2}^{q-1} (d_{S_i}(x_q) + d_{S_i^+}(x_1)) \leqslant \sum_{i \in A_1 \cup A_2} |S_i| + \sum_{i \in A_3} (|S_i^+| + 1)$$

$$= \sum_{i \in A_1 \cup A_2 \cup A_3} |S_i| + |A_3|$$

$$\leqslant \sum_{i\in A_1\cup A_2\cup A_3} |S_i| + (q-2-|A_1\cup A_2\cup A_3|)$$

$$\leqslant \sum_{i=2}^{q-1} |S_i|.$$

The vertices x_1 and x_q satisfy the following inequalities:

$$|C'| + \varepsilon \leqslant d_{C'}(x_q) + d_{C'}(x_1)$$

$$= d_{S_1}(x_q) + d_{S_1^+}(x_1) + d_{S_q'}(x_q) + d_{S_q'^+}(x_1) + \sum_{i=2}^{q-1}(d_{S_i}(x_q) + d_{S_i^+}(x_1))$$

$$\leqslant |S_1| + |S_q'| + \sum_{i=2}^{q-1} |S_i| = |C'|.$$

This implies $\varepsilon = 0$ and so $x_q^+ = x_1$ in C. Therefore, $C' = C$, $G' = G$ and $d_C(x_q) + d_C(x_1) = |C|$. By the choice of x_q and x_1, no pair of consecutive S-vertices has degree sum in C greater than $|C|$. In particular, $d_C(x_{l-1}) + d_C(x_l) \leqslant |C|$. But by (*), $d_C(x_q) + d_C(x_{l-1}) \geqslant |C|$ and $d_C(x_1) + d_C(x_l) \geqslant |C|$. So we get $d_C(x_q) = d_C(x_l)$, $d_C(x_1) = d_C(x_{l-1})$, and thus $x_l = x_{l-1}^+$ since x_{l-1} and x_l can play the same role as x_q and x_1. Considering now the S-consecutive pairs x_1, x_2 and x_l, x_{l+1}, we can show in the same way that both pairs are consecutive in C with degree sum in C equal to $|C|$, which implies $d_C(x_1) = d_C(x_{l+1})$ and $d_C(x_2) = d_C(x_l)$. More generally, every vertex of C belongs to S, every two consecutive vertices in C have degree sum in C equal to $|C|$, S-vertices with odd (even, respectively) subscript have degree in C equal to $d_C(x_1)$ ($d_C(x_q)$, respectively) and thus $|S|$ is even. We will use the following theorem.

Theorem 4 (Schmeichel and Hakimi [7]). *Let G be a graph with a hamiltonian cycle $C := x_1x_2\ldots x_nx_1$ with $n \geqslant 3$. Suppose $d(x_1) + d(x_n) \geqslant n$, with say $d(x_1) \leqslant d(x_n)$. Then*
(i) *G is pancyclic or*
(ii) *G is bipartite or*
(iii) *G contains cycles of all lengths except an $(n-1)$-cycle.*
Moreover, if (iii) holds, then $d(x_{n-2}), d(x_{n-1}), d(x_2), d(x_3) < n/2$.

We know that case (iii) could not happen in our proof because every two consecutive vertices in C have degree sum in C equal to $|C|$. Since $G[S]$ is not pancyclic, we conclude that $G[S]$ is the complete bipartite graph $K_{|S|/2,|S|/2}$.

If $S = V(G)$, then G is isomorphic to $K_{n/2,n/2}$. If not, any two nonadjacent S-vertices belong to the same class of $K_{|S|/2,|S|/2}$, and have degree sum in R equal to at least $|R|$. Moreover, if $|S| \geqslant 6$, there must be two nonadjacent S-vertices u and v with a common neighbor w in R. Clearly $G[S]$ contains cycles of all even lengths between 4 and $|S|$. Since $G[S]$ contains all paths of endvertices u and v of all odd lengths between 3 and $|S|-1$, then these paths together with the vertex w give cycles containing exactly all odd numbers between 3 and $|S|-1$ of S-vertices. Thus G is S-pancyclable, which contradicts the assumption. Therefore $|S| = 4$ and $G[S]$ is isomorphic to an induced cycle $x_1x_2x_3x_4x_1$. The vertices x_1 and x_3, and similarly x_2 and x_4, have no common neighbors for otherwise G would contain cycles of S-length 3 and would be S-pancyclable. It is then easy to see that G has the structure described in Theorem 3. □

Going back through the above proof, we observe that we also have obtained the following result, analogous to the one proved in [1].

Theorem 5. *Let G be a graph of order n, S a subset of $V(G)$ such that S is cyclable in G, and let C be a shortest cycle through all vertices of S. If $d(x)+d(y)\geqslant n+1$ for some pair of S-vertices x and y consecutive in C, then G is S-pancyclable.*

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